On correlations and mutual entropy in quantum composed systems

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We study the correlations of classical and quantum systems from the information theoretical points of view. We analyze a simple measure of correlations based on entropy (such measure was already investigated as a *degree of entanglement* by Belavkin, Matsuoka and Ohya).

§1. Introduction

There are two fundamental problems in quantum information theory: a classification of quantum states of composite systems and description of correlations encoded into such states.

In classical description of physical compound systems we assume that complete information about the whole system is equivalent to the complete knowledge about properties of all its subsystems. This is no longer true in the quantum case. In the quantum world, there exist states of compound systems for which we have the complete information, while our knowledge about the subsystems can be expressed only in stochastic way. Such non-classical property of quantum system is due to the difference of two types of probability theory. In classical probability theory the classical correlation of a physical compound system can be represented by a joint probability measure with marginal probability measures which are corresponding to each subsystem. In quantum case a quantum state of a composite system, i.e. a compound density operator describes a correlation between its marginal states. However, in quantum system, it is known that the joint probability and the conditional probability do not generally exist, which is an essential difference from classical system.[36),41)] The typical example of such difference is the existence of entangled states in quantum system.

An entangled state of a composed quantum system is defined as a state which can not be represented as a convex sum $\sum_k \lambda_k \rho_k \otimes \sigma_k$. States which are not entangled are called separable. However the mathematical characterization of entanglement is not yet fully understood except for some simple cases, for example pure states or very low dimensional mixed states. Such correlated states have been considered in several contexts in the field of quantum information and quantum probability such as quantum measurement and filtering,[9),10)] quantum compound states[34),35)] and lifting.[1),4)] So that one have to deal with not only the correlation of entangled state but also that of separable state more carefully. From this point of view to give the proper classification of quantum states is of primary importance. Hence, one needs suitable measure of *quantum correlations.*

This paper is organized as follows: In Section 2 we give the simple review of the classical information theory and make clear the crucial points of classical theory. In Section 3 we analyze three approaches to quantum entanglement based on PPT condition,[38),24)] an entanglement map and a quantum conditional probability operator (QCPO for short).[5)] Some relations between these approaches are derived.[5),20),27),30)]

In Section 4, we review the two types of quantum entropy, so called the quantum mutual entropy and the quantum conditional entropy,[11),12),14),21),23)] and such entropies are applied to some examples of compound states. On the basis of above observation we introduce the order of

quantum correlation by using the symmetrized conditional entropy called the degree of entanglement.[20),32)] Section 5 and 6 provide some examples of circulant state[16),17)] which can easily compute degree of entanglement. Moreover, since they possess the same marginal states (maximally mixed) one can estimate the order of correlation for such class of quantum states. Interestingly it turned out that there exists a class of separable states which have stronger correlation than entangled states.[16)] Finally we discuss the difference between classical and quantum correlations in Section 7. We remark that those argument in above can be formulated at general C*-algebraic setting.[11),12),27),30),31)]

Throughout the paper, we use calligraphic symbols $\mathcal{H}$, $\mathcal{K}$ for complex separable Hilbert spaces and denote the set of the bounded operators and the set of all states on $\mathcal{H}$ by $\mathbf{B}(\mathcal{H})$ and $\mathbf{S}(\mathcal{H})$, respectively. In the d-dimensional Hilbert space, the standard basis is denoted by $\{e_0, e_1, \cdots, e_{d-1}\}$ and the inner product is denoted by $\langle\cdot,\cdot\rangle$. We write e_{ij} for $|e_i\rangle\langle e_j|$. Given any state θ on the tensor product Hilbert space $\mathcal{H}\otimes\mathcal{K}$, we denote by $\mathrm{Tr}_{\mathcal{K}}\theta$ the partial trace of θ with respect to $\mathcal{K}$.

§2. Basic terminology on classical information theory

In classical description of a physical compound system its correlation can be represented by a joint probability measure or a conditional probability measure. In classical information theory we have proper criteria to estimate such correlation, which are so-called the mutual entropy and the conditional entropy given by Shannon.[39)] Here we review Shannon's entropies briefly.

Let $X = \{x_i\}_{i=1}^n$ and $Y = \{y_j\}_{j=1}^m$ be random variables with probability distributions $P = \{p_i\}_i$ and $Q = \{q_j\}_j$, and let $X\otimes Y := (X\times Y, R)$, $(R = \{r_{i,j}\}_{i,j})$ be their compound system. The joint probability $r_{i,j}$ is represented by

$$r_{i,j} = p(i\mid j)q_j = p(j\mid i)p_i, \tag{2.1}$$

where $p(i\mid j)$ (resp. $p(j\mid i)$) is a conditional probability of X (resp. Y) under the observation of Y (resp. X). Then the definition of mutual entropy $I(X,Y)$ and conditional entropies $S(X\mid Y)$, $S(Y\mid X)$ are given by

$$I(X,\,Y) := \sum_{i,j} r_{i,j}\log\frac{r_{i,j}}{p_i q_j}, \tag{2.2}$$

$$S(X\mid Y) := -\sum_j q_j\sum_i p(i\mid j)\log p(i\mid j), \tag{2.3}$$

$$S(Y\mid X) := -\sum_i p_i\sum_j p(j\mid i)\log p(j\mid i). \tag{2.4}$$

Using (2.1), we can easily check that the following relations

$$I(X,\,Y) = S(X) + S(Y) - S(X\otimes Y), \tag{2.5}$$

$$S(X\mid Y) = S(X\otimes Y) - S(Y) = S(X) - I(X,\,Y), \tag{2.6}$$

$$S(Y\mid X) = S(X\otimes Y) - S(X) = S(Y) - I(X,\,Y). \tag{2.7}$$

where $S(X) = -\sum_i p_i\log p_i$ and $S(X\otimes Y) = -\sum_{ij} r_{ij}\log r_{ij}$. From the above definitions and relations such entropies can be recognized as measures for the correlation between two marginal probabilities. The mutual entropy $I(X,\,Y)$ is given as the relative entropy of the correlated joint probability r_{ij} and the non-correlated one $p_i q_j$. On the other hand, the conditional entropy is given by the average of the entropy of conditional probability.

The mutual entropy has another role in classical information theory, namely the role as the measure of transmitted information through a channel. Indeed, the mutual entropy $I(X,\,Y)$ can be

represented by using the conditional probability $p(j \mid i)$ and the probability p_i as

$$\begin{aligned} I(X,\ Y) &:= \sum_{i,j} r_{ij} \log \frac{r_{ij}}{p_i q_j} \\ &= \sum_{i,j} p(j \mid i) p_i \log \frac{p(j \mid i) p_i}{p_i q_j} \\ &= \sum_{i,j} p(j \mid i) p_i \log \frac{p(j \mid i)}{\sum_k p(j \mid k) p_k}. \end{aligned} \tag{2.8}$$

Now, let us denote a transition probability matrix $(p(j \mid i))$ by $\Lambda^* = (\Lambda^*_{j,i})$ and call it a channel. We call the probability distribution $P = \{p_i\}$ an input state, and the probability distribution $Q = \{q_j\}$ – an output state – given by

$$q_j = \sum_i \Lambda^*_{j,i}\, p_i =: (\Lambda^* P)_j\ . \tag{2.9}$$

Finally, let us replace the notation of the mutual entropy $I\,(X, Y)$ with $I\,(P; \Lambda^*)$. This mutual entropy $I(P; \Lambda^*)$ can be regarded as a measure of transmitted information through the channel Λ^* from an input state P to an output state $Q = \Lambda^* P$. From this point of view the amount of transmitted information through the channel corresponds to the strength of correlation between input system and output system which are connected to each other via the channel. In other words the transition (or conditional) probability $p(j \mid i)$ also represents the correlation between the input state P and the output state $Q = \Lambda^* P$. Then the conditional entropy $S\,(X \mid Y)$ defines the amount of information about the input system which can not be transmitted to the output system through the channel Λ^*. Actually the conditional entropy $S\ (X|Y)$ in (2.6) is represented by

$$S\ (X|Y) = S\ (P) - I\,(P; \Lambda^*)\,. \tag{2.10}$$

Let us note the important property of mutual entropy displayed by the following inequality (called the Shannon's fundamental inequality):

$$0 \le I(P; \Lambda^*) \le \min\{S\,(P),\ S\,(\Lambda^* P)\}. \tag{2.11}$$

It is natural that the transmitted information is less than the information of the input system. So if the whole information of the input system is transmitted through the channel, then such communication system has a perfect correlation in the sense of transmitted information. In other words, such channel can be recognized as a noiseless channel.

Let us summarize the crucial points of classical information theory:

1. The mutual entropy is a proper measure of correlation as the transmitted information through the channel. This interpretation can be done due to the relation between the joint probability and the conditional probability in (2.1).
2. The mutual entropy satisfies the Shannon's fundamental inequality in (2.11).

§3. Separable states, PPT states and entanglement maps

We start with recalling the two kind of notions of positivity, so called, complete positivity and complete co-positivity. A linear map $\chi : \mathbf{B}(\mathcal{K}) \to \mathbf{B}(\mathcal{H})$ is said to be completely positive (CP, for short) if, for any $n \in \mathbb{N}$, the map

$$\chi_n : M_n(\mathbb{C}) \otimes \mathbf{B}(\mathcal{K}) \longrightarrow M_n(\mathbb{C}) \otimes \mathbf{B}(\mathcal{H}), \quad (a_{i,j})_{i,j} \longmapsto (\chi(a_{i,j}))_{i,j} \tag{3.1}$$

is positive. Here $M_n(\mathbb{C})$ stands for $n \times n$ matrices with entries in $\mathbb{C}$. A linear map $\chi : \mathbf{B}(\mathcal{K}) \to \mathbf{B}(\mathcal{H})$ is said to be completely copositive (CCP, for short) if $\tau \circ \chi$, where τ stands for transposition, is CP.

Let us consider two quantum systems $(\mathcal{H}, \mathbf{B}(\mathcal{H}), \mathbf{S}(\mathcal{H}))$ and $(\mathcal{K}, \mathbf{B}(\mathcal{K}), \mathbf{S}(\mathcal{K}))$. Then for the composite system $(\mathcal{H}_{\text{total}}, \mathbf{B}, \mathbf{S})$ one has

$$\mathcal{H}_{\text{total}} \equiv \mathcal{H} \otimes \mathcal{K} \ , \ \mathcal{A} \equiv \mathbf{B}\,(\mathcal{H} \otimes \mathcal{K}) \ , \ \mathbf{S} \equiv \mathbf{S}\,(\mathcal{A}) = \mathbf{S}\,(\mathbf{B}(\mathcal{H}) \otimes \mathbf{B}(\mathcal{K}))\,.$$

Notice that

$$\mathbf{S} \supset \text{conv}(\mathbf{S}(\mathcal{H}) \otimes \mathbf{S}(\mathcal{K})).$$

It was mentioned that $\text{conv}(\mathbf{S}(\mathcal{H}) \otimes \mathbf{S}(\mathcal{K}))$ are called separable states and the subset of states $\mathbf{S}\backslash\text{conv}(\mathbf{S}(\mathcal{H}) \otimes \mathbf{S}(\mathcal{K}))$ is called the set of entangled states, i.e. $\mathbf{S} = \mathbf{S}_{\text{ENT}} \cup \mathbf{S}_{\text{SEP}}$. We are interested in the special subset of $\mathbf{S}$,

$$\mathbf{S}_{\text{PPT}} \equiv \{\omega \in \mathbf{S}\,|\,\omega \circ (1 \otimes \tau) \in \mathbf{S}\}\,. \tag{3.2}$$

Such states are called partial positive transposition states (PPT).[24),38)] Clearly,

$$\mathbf{S} \supset \mathbf{S}_{\text{PPT}} \supset \mathbf{S}_{\text{SEP}}.$$

In general the PPT condition is not sufficient for separability. However, one has[24)]

Theorem 3.1 *For the pairs of Hilbert spaces* $(\mathcal{H}, \mathcal{K}) = (\mathbb{C}^2, \mathbb{C}^2), (\mathbb{C}^2, \mathbb{C}^3), (\mathbb{C}^3, \mathbb{C}^2)$, *a state* $\omega \in \mathbf{S}$ *is separable if and only if it is* PPT.

For a normal state $\omega \in \mathbf{S}$ there exists a density matrix θ such that

$$\omega(a \otimes b) = \text{Tr}(a \otimes b)\theta, \tag{3.3}$$

for any $a \in \mathbf{B}(\mathcal{H})$, $b \in \mathbf{B}(\mathcal{K})$. By using the linear map $\phi : \mathbf{B}(\mathcal{K}) \to \mathbf{B}(\mathcal{H})$ given by $\phi(b) := \text{Tr}_{\mathcal{K}}(1_{\mathcal{H}} \otimes b)\theta$, and its dual $\phi^* : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{K}), a \mapsto \text{Tr}_{\mathcal{H}}(a \otimes 1_{\mathcal{K}}\theta)$, the state ω can be written as

$$\omega(a \otimes b) = \text{Tr}_{\mathcal{H}}\ a\phi(b) = \text{Tr}_{\mathcal{K}}\ \phi^*(a)b. \tag{3.4}$$

It is clear that the marginal densities of θ are given by

$$\text{Tr}_{\mathcal{K}}\theta = \phi(1_{\mathcal{K}}) \in \mathbf{B}(\mathcal{H}), \quad \text{Tr}_{\mathcal{H}}\theta = \phi^*(1_{\mathcal{H}}) \in \mathbf{B}(\mathcal{K}). \tag{3.5}$$

Belavkin and Ohya observed[11),12)] that such maps can be reconstructed by the Hilbert-Schmidt operator, which is called the entangling operator, and they showed that both ϕ and ϕ^* are CCP, but not always CP. For example, if ω is a pure entangled state, then its entanglement map is not CP.

Definition 3.2 *A CCP map* $\phi : \mathbf{B}(\mathcal{K}) \to \mathbf{B}(\mathcal{H})_*$ *normalized as* $\text{Tr}_{\mathcal{H}}\phi(1_{\mathcal{K}}) = 1$ *is called the entanglement map from* $\phi^*(1_{\mathcal{H}}) \in \mathbf{B}(\mathcal{K})$ *to* $\phi(1_{\mathcal{K}}) \in \mathbf{B}(\mathcal{H})$.

The density operator θ_ϕ corresponding to the entanglement map ϕ with its marginals $\phi^*(1_{\mathcal{H}})$ and $\phi(1_{\mathcal{K}})$ can be represented by

$$\theta_\phi = \sum_{i,j} \phi(|e_j\rangle\langle e_i|) \otimes |e_i\rangle\langle e_j| \tag{3.6}$$

(see,[11)12)]). The relation $\phi \longleftrightarrow \theta_\phi$ is usually called Choi-Jamiołkowski isomorphism.

Lemma 3.3 *A linear map* $\phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ *is CCP if and only if* $\theta_\phi \geq 0$. *Clearly,* ϕ *is CP if and only if* $\phi \circ \tau$ *is CCP.*

Due to Lemma 3.3, we have the following criterion.

Theorem 3.4 [27),30)] *A state* θ_ϕ *is a PPT state if and only if its entanglement map* ϕ *is CP.*

Recently, Kossakowski et al. gave the following construction of compound density operators in.[5)] For $\theta \in \mathbf{S}(\mathcal{H} \otimes \mathcal{K})$, we define the bounded operator

$$\pi_\theta := (\rho^{-\frac{1}{2}} \otimes 1_{\mathcal{K}})\,\theta\,(\rho^{-\frac{1}{2}} \otimes 1_{\mathcal{K}}), \tag{3.7}$$

where $\rho := \text{Tr}_{\mathcal{K}}\theta$. It is verified that π_θ satisfies

$$\pi_\theta \geq 0, \tag{3.8}$$

$$\mathrm{Tr}_{\mathcal{K}}\pi_\theta = 1_{\mathcal{H}} \in \mathbf{B}(\mathcal{H}). \tag{3.9}$$

We assume that ρ is a faithful state, i.e. $\rho > 0$. It follows from (3.8) and (3.9) that the operator π_θ is the quantum analogue of a classical conditional probability. Indeed, if $\mathbf{B}(\mathcal{H} \otimes \mathcal{K})$ is replaced by commutative algebra, then π_θ coincides with a classical conditional probability.

Definition 3.5 *An operator $\pi \in \mathbf{B}(\mathcal{H} \otimes \mathcal{K})$ is called the quantum conditional probability operator (QCPO, for short) if π satisfies condition (3.8) and (3.9).*

It is easy to verify that any CP unital map $\mathbf{B}(\mathcal{H} \otimes \mathcal{K})$ can be represented in the form

$$\pi_\varphi = \sum_{k,\ell}^{n} \varphi(|e_k\rangle\langle e_\ell|) \otimes |e_k\rangle\langle e_\ell|. \tag{3.10}$$

From Lemma 3.3 and unitality of φ, it follows that π_φ satisfies conditions (3.8) and (3.9). For a given π_φ and any marginal state $\rho \in \mathbf{S}(\mathcal{H})$, one can construct a compound state as

$$\theta_\varphi = \sum_{k,\ell}^{n} \rho^{\frac{1}{2}}\, \varphi\,(|e_k\rangle\langle e_\ell|)\rho^{\frac{1}{2}} \otimes |e_k\rangle\langle e_\ell|. \tag{3.11}$$

It is clear that θ_φ is a PPT state if and only if the map φ is a CCP. There exists a simple relation between the density operator θ_ϕ in (3.6) and the QCPO π_φ in (3.10) due to the following decomposition of the entanglement map ϕ.

Lemma 3.6 [13] *Every entanglement map ϕ with $\phi(1_{\mathcal{K}}) = \rho$ has a decomposition*

$$\phi\,(\cdot) = \rho^{\frac{1}{2}}\varphi \circ \tau\,(\cdot)\,\rho^{\frac{1}{2}}, \tag{3.12}$$

where φ is a CP unital map to be found as a unique solution to

$$\varphi(\cdot) = \rho^{-\frac{1}{2}}\phi \circ \tau(\cdot)\rho^{-\frac{1}{2}}. \tag{3.13}$$

Theorem 3.7 [20] *If a compound state θ_ϕ given by (3.6) has a faithful marginal state $\rho = \phi(1_{\mathcal{K}})$, then θ_ϕ is represented by*

$$\theta_\phi = (\rho^{\frac{1}{2}} \otimes 1_{\mathcal{K}})\,\pi_\phi\,(\rho^{\frac{1}{2}} \otimes 1_{\mathcal{K}}), \tag{3.14}$$

where $\pi_\phi = \sum_{k,\ell} \rho^{-\frac{1}{2}}\,\phi\,(|e_\ell\rangle\langle e_k|)\,\rho^{-\frac{1}{2}} \otimes |e_k\rangle\langle e_\ell|$.

The relation (3.14) can be regarded as a quantum analogue of the fundamental probability relation (2.1). We call it the quantum density relation.

§4. Quantum mutual entropy and quantum conditional entropy

We extend the classical mutual entropy to the quantum system using the Umegaki relative entropy.[40]

Definition 4.1 [11),12),14),21)] *For any entanglement map $\phi : \mathbf{B}(\mathcal{K}) \to \mathbf{B}(\mathcal{H})$ with $\rho = \phi(1_{\mathcal{K}})$ and $\sigma = \phi^*(1_{\mathcal{H}})$, the quantum mutual entropy $I_\phi(\rho, \sigma)$ is defined by*

$$I_\phi(\rho,\, \sigma) := S\,(\theta_\phi, \rho \otimes \sigma) = \mathrm{Tr}\theta_\phi(\log\theta_\phi - \log(\rho \otimes \sigma)), \tag{4.1}$$

where $S\,(\cdot,\,\cdot)$ is the Umegaki relative entropy.

The quantum mutual entropy $I_\phi(\rho,\, \sigma)$ can be computed as follows:

$$\begin{aligned} I_\phi(\rho, \sigma) &= \mathrm{Tr}\theta_\phi(\log\theta_\phi - (\log\rho \otimes \sigma)) \\ &= \mathrm{Tr}\theta_\phi(\log\theta_\phi - \log(\rho \otimes 1_{\mathcal{K}}) - \log(1_{\mathcal{H}} \otimes \sigma)) \\ &= -S\,(\theta_\phi) - \mathrm{Tr}_{\mathcal{K}}\theta_\phi \log(\rho \otimes 1_{\mathcal{K}}) - \mathrm{Tr}_{\mathcal{H}}\theta_\phi \log(1_{\mathcal{H}} \otimes \sigma) \\ &= S\,(\rho) + S\,(\sigma) - S\,(\theta_\phi). \end{aligned} \tag{4.2}$$

The above relation (4.2) is a quantum analog of (2.5). Using this decomposition (4.2) one also defines the quantum conditional entropies as generalizations of (2.6), (2.7):[11),12),14),23)]

$$S_\phi(\sigma \mid \rho) := S(\sigma) - I_\phi(\rho, \sigma) = S(\theta_\phi) - S(\rho), \tag{4.3}$$

$$S_\phi(\rho \mid \sigma) := S(\rho) - I_\phi(\rho, \sigma) = S(\theta_\phi) - S(\sigma). \tag{4.4}$$

Example 4.2 *(Product state) For entanglement maps $\phi(b) = \rho \mathrm{Tr}_{\mathcal{K}} \sigma b$ and $\phi^*(a) = \sigma \mathrm{Tr}_{\mathcal{H}} \rho a$, we have $\theta_\phi = \rho \otimes \sigma$ and hence*

$$I_\phi(\rho, \sigma) = 0\ , \quad S_\phi(\sigma \mid \rho) = S(\sigma)\ , \quad S_\phi(\rho \mid \sigma) = S(\rho), \tag{4.5}$$

which shows that a product state has no correlations at all.

For the state θ induced from the entangle map ϕ, the quantum mutual entropy $I_\phi(\rho, \sigma)$ measures the correlation between θ and the product state given by its marginal densities. In quantum system we have two types of correlated states. The first one is a separable correlated state, and second one is an entangled correlated state.

Example 4.3 *(Separable correlated state) For the entanglement map given by*

$$\phi(b) = \sum_i \lambda_i \rho_i \mathrm{Tr} \sigma_i b, \quad \phi^*(a) = \sum_i \lambda_i \sigma_i \mathrm{Tr} \rho_i a, \quad \Big(\sum_i \lambda_i = 1,\ \lambda_i \geq 0\, \forall i \Big),$$

the state θ_ϕ can be written in the form

$$\theta_\phi = \sum_i \lambda_i \rho_i \otimes \sigma_i, \tag{4.6}$$

with $\rho = \phi(1_{\mathcal{K}}) = \sum_i \lambda_i \rho_i$, $\sigma = \phi^*(1_{\mathcal{H}}) = \sum_i \lambda_i \sigma_i$. Then, we have the following inequalities.[3),11),12)]

$$0 \leq I_\phi(\rho, \sigma) \leq \min\{S(\rho), S(\sigma)\}, \tag{4.7}$$

$$S_\phi(\sigma \mid \rho) \geq 0, \quad S_\phi(\rho \mid \sigma) \geq 0. \tag{4.8}$$

Example 4.4 *(Separable perfectly correlated state) Let $\{e_i\}_i$ and $\{f_j\}_j$ be the CONSs in $\mathcal{H}$ and $\mathcal{K}$, respectively. For the entanglement map given by*

$$\phi(b) = \sum_i \lambda_i |e_i\rangle\langle e_i| \langle f_i, b f_i\rangle, \quad \phi^*(a) = \sum \lambda_i |f_i\rangle\langle f_i| \langle e_i, a e_i\rangle,$$

the state θ_ϕ can be written in the form

$$\theta_\phi = \sum \lambda_i |e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|$$

with $\rho = \phi(1_{\mathcal{K}}) = \sum \lambda_i |e_i\rangle\langle e_i|$, $\sigma = \phi^*(1_{\mathcal{H}}) = \sum_i \lambda_i |f_i\rangle\langle f_i|$. Then,

$$I_\phi(\rho, \sigma) = S(\rho) + S(\sigma) - S(\theta_\phi) = S(\rho), \tag{4.9}$$

$$S_\phi(\sigma \mid \rho) = S_\phi(\rho \mid \sigma) = 0, \tag{4.10}$$

where $S(\rho) = S(\sigma) = S(\theta_\phi) = -\sum \lambda_i \log \lambda_i$. This correlation corresponds to a perfect correlation in the classical scheme.

Example 4.5 *(Pure entangled state) Let $\{\lambda_i\}$ be the sequence of complex numbers satisfying $\sum_i |\lambda_i|^2 = 1$. For entanglement mappings*

$$\phi(b) = \sum_{i,j} \lambda_i \overline{\lambda}_j |e_i\rangle\langle e_j| \langle f_j,\ b f_i\rangle, \quad \phi^*(a) = \sum_{i,j} \lambda_i \overline{\lambda}_j |f_i\rangle\langle f_j| \langle e_j,\ a e_i\rangle,$$

the state θ_ϕ can be written in the form

$$\theta_\phi = \sum_{i,j} \lambda_i \overline{\lambda}_j |e_i\rangle\langle e_j| \otimes |f_i\rangle\langle f_j| = \big|\Psi\rangle\langle\Psi\big|,$$

where

$$\left|\Psi\right\rangle = \sum_i \lambda_i\, e_i \otimes f_i.$$

One has $\rho = \phi(1_{\mathcal{K}}) = \sum_i |\lambda_i|^2 |e_i\rangle\langle e_i|$ *and* $\sigma = \phi^*(1_{\mathcal{H}}) = \sum_i |\lambda_i|^2 | f_i\rangle\langle f_i|$*. Then,*

$$\begin{aligned} I_\phi(\rho, \sigma) &= S(\rho) + S(\sigma) - S(\theta_\phi) \\ &= 2S(\rho) > \min\{S(\rho), S(\sigma)\}, \end{aligned} \tag{4.11}$$

$$S_\phi(\sigma \mid \rho) = S_\phi(\rho \mid \sigma) = -S(\rho) < 0, \tag{4.12}$$

where $S(\rho) = S(\sigma) = -\sum_i |\lambda_i|^2 \log |\lambda_i|^2$.

As is mentioned in Section 2, the classical mutual entropy always satisfies the Shannon's fundamental inequality, i.e. it is always smaller than its marginal entropies, and the conditional entropy is always positive. Note that separable state has the same property. It is no longer true for pure entangled states.

We introduce another measure for correlation of compound states.[11),12),20),32)]

Definition 4.6 [11),12),32),20)] *For the entanglement map* $\phi : \mathbf{B}(\mathcal{K}) \to \mathbf{B}(\mathcal{H})$*, we define*

$$D_\phi(\rho,\, \sigma) := -\frac{1}{2}\left\{S_\phi(\sigma \mid \rho) + S_\phi(\rho \mid \sigma)\right\} = \frac{1}{2}(S(\rho) + S(\sigma)) - S(\theta_\phi)\,. \tag{4.13}$$

Note, that $D^E_\phi(\rho,\, \sigma) = -D_\phi(\rho,\, \sigma)$ is called degree of entanglement.

Proposition 4.7 [2),32)] *If* θ_ϕ *is a pure state, then the following statements hold:*

1. θ_ϕ *is entangled state if and only if* $D_\phi(\rho,\, \sigma) > 0$.
2. θ_ϕ *is separable state if and only if* $D_\phi(\rho, \sigma) = 0$.

It is well-known that if θ is a PPT state, then

$$S(\theta) - S(\rho) \geq 0, \quad S(\theta) - S(\sigma) \geq 0, \tag{4.14}$$

where ρ and σ are the marginal states of θ.[42)]

Proposition 4.8 *If* θ_ϕ *is a mixed PPT state, then*

$$D_\phi(\rho,\, \sigma) \leq 0. \tag{4.15}$$

Suppose now that we have two entanglement mappings $\phi_k : \mathbf{B}(\mathcal{K}) \to \mathbf{B}(\mathcal{H})$, $(k = 1, 2)$ such that $\phi_1(1_{\mathcal{K}}) = \phi_2(1_{\mathcal{K}})$ and $\phi_1^*(1_{\mathcal{H}}) = \phi_2^*(1_{\mathcal{H}})$

Definition 4.9 ϕ_1 *is said to have stronger correlation than* ϕ_2 *if*

$$D_{\phi_1}(\rho,\, \sigma) > D_{\phi_2}(\rho,\, \sigma). \tag{4.16}$$

Several measures based entropic quantities are discussed also by Cerf and Adami,[14)] Horodecki,[23)] Henderson and Vedral,[22)] Groisman et al.[21)]

§5. The quantum correlation for circulant states

5.1. *A circulant state*

We start this section by recalling the definition of circulant state introduced in[17)] (see also[18)]). Consider the finite-dimensional Hilbert space $\mathbb{C}^d$ $(d \in \mathbb{N})$ with the standard basis $\{e_0,\, e_1,\, \cdots,\, e_{d-1}\}$. Let Σ_0 be the subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$ generated by $e_i \otimes e_i$ $(i = 0,\, 1,\, \cdots,\, d-1)$:

$$\Sigma_0 = \mathrm{span}\{e_0 \otimes e_0,\; e_1 \otimes e_1,\; \cdots, e_{d-1} \otimes e_{d-1}\}. \tag{5.1}$$

For any non-negative integer α, we define the operator S^α on $\mathbb{C}^d$ by

$$e_k \longmapsto e_{k+\alpha\,(\mathrm{mod}\, d)}, \quad (k = 0,\, 1,\, \cdots,\, d-1),$$

and denote by Σ_α the image of Σ_0 by $\mathrm{I}_d \otimes S^\alpha : \Sigma_\alpha = (\mathrm{I}_d \otimes S^\alpha)\Sigma_0$. It turns out that Σ_α and Σ_β ($\alpha \neq \beta$) are orthogonal to each other and

$$\mathbb{C}^d \otimes \mathbb{C}^d \cong \Sigma_0 \oplus \Sigma_1 \oplus \cdots \oplus \Sigma_{d-1}. \tag{5.2}$$

This decomposition is called a circulant decomposition.

Let $\rho_0, \rho_1, \cdots, \rho_{d-1}$ be positive $d \times d$ matrices with entries in $\mathbb{C}$ which satisfy

$$\mathrm{tr}(\rho_0 + \cdots + \rho_{d-1}) = 1. \tag{5.3}$$

For each matrix ρ_α ($\alpha = 0, 1, \cdots, d-1$), we define the new linear operator $\rho_\alpha^\sharp$ on $(\mathbb{C}^d)^{\otimes 2}$ as

$$\rho_\alpha^\sharp = \sum_{i,j=0}^{d-1} \langle e_i, \rho_\alpha e_j \rangle e_{ij} \otimes S^\alpha e_{ij} (S^\alpha)^*, \tag{5.4}$$

where e_{ij} means $|e_i\rangle\langle e_j|$. Since $S^k e_{ij}(S^k)^* = e_{i+k,\, j+k}$, we note that $\rho_\alpha^\sharp$ can be also written as

$$\rho_\alpha^\sharp = \sum_{i,j=0}^{d-1} \langle e_i, \rho_\alpha e_j \rangle e_{ij} \otimes e_{i+\alpha,\, j+\alpha}. \tag{5.5}$$

One can easily check that the sum of these operators

$$\rho^\sharp = \sum_{\alpha=0}^{d-1} \rho_\alpha^\sharp \tag{5.6}$$

defines a density matrix on $(\mathbb{C}^d)^{\otimes 2}$. For further details of circulant states we refer to Refs.[17),18)]

5.2. *A family of Horodecki states*

In this and all subsequent sections, we assume that $\mathcal{H} = \mathcal{K} = \mathbb{C}^3$, so, $\mathbf{B}(\mathcal{H}) = \mathbf{B}(\mathcal{K}) = M_3(\mathbb{C})$. For any $\alpha \in [2, 5]$, we define a linear map $\phi_\alpha : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ as

$$\phi_\alpha = \frac{2}{7}\phi_0 + \frac{\alpha}{7}\phi_1 + \frac{5-\alpha}{7}\phi_2, \tag{5.7}$$

where

$$\phi_0(a) = \frac{1}{3}\sum_{i,j=0}^{2} e_{ij} \left\langle e_j, a e_i \right\rangle \ , \ \ \phi_1(a) = \frac{1}{3}\sum_{i=0}^{2} e_{ii} \left\langle S^1 e_i, a S^1 e_i \right\rangle \ , \ \ \phi_2(a) = \frac{1}{3}\sum_{i=0}^{2} e_{ii} \left\langle S^2 e_i, a S^2 e_i \right\rangle \ .$$

Notice that ϕ_0 is a pure maximal entanglement map and ϕ_1 and ϕ_2 are corresponding to a separable perfectly correlated state (see Example 4.4).

The marginal states ρ and σ are calculated easily as

$$\phi_\alpha(1) = \rho = \sigma = \frac{1}{3}\Big(e_{00} + e_{11} + e_{22} \Big), \tag{5.8}$$

and the compound state $\theta_1(\alpha)$ induced from ϕ_α is given by

$$\begin{aligned}\theta_1(\alpha) = \sum_{i,j} \phi_\alpha(e_{ji}) \otimes e_{ij} = \frac{2}{21}\sum_{i,j} e_{ij} \otimes e_{ij} + \frac{\alpha}{21}\Big(e_{00} \otimes e_{11} + e_{11} \otimes e_{22} + e_{22} \otimes e_{00}\Big) \\ + \frac{5-\alpha}{21}\Big(e_{00} \otimes e_{22} + e_{11} \otimes e_{00} + e_{22} \otimes e_{11}\Big).\end{aligned} \tag{5.9}$$

The eigenvalues of $\theta_1(\alpha)$ are calculated as 0, $\frac{2}{7}$, $\frac{\alpha}{7}$ and $\frac{5-\alpha}{7}$. So, we get

$$D(\theta_1(\alpha)) = -\frac{2}{7}\log\frac{2}{7} - \frac{\alpha}{7}\log\frac{\alpha}{7} - \frac{5-\alpha}{7}\log\frac{5-\alpha}{7} - \frac{2}{7}\log 3. \tag{5.10}$$

where $D(\theta_1(\alpha)) := D_{\phi_\alpha}(\rho,\rho)$. We note that $\theta_1(\alpha)$ reconstructs the family of states analyzed in.[25)] One proves

Theorem 5.1 [25)] *The states of $\theta_1(\alpha)$ can be classified by the values of α.*

1. *$\theta_1(\alpha)$ is separable if and only if $2 \le \alpha \le 3$;*
2. *$\theta_1(\alpha)$ is both entangled and PPT if and only if $3 < \alpha \le 4$;*
3. *$\theta_1(\alpha)$ is NPT if and only if $4 < \alpha \le 5$.*

Due to this theorem, one can find that the $D(\theta_1(\alpha))$ admits a natural order. That is, the correlation for an entangled state is stronger than a separable one. In addition we know also that the correlation for a NPT state is stronger than a PPT one. The graph of $D(\theta_1(\alpha))$ is shown in Fig. 1. This quantity is always negative whenever $\alpha \in (4, 5]$.

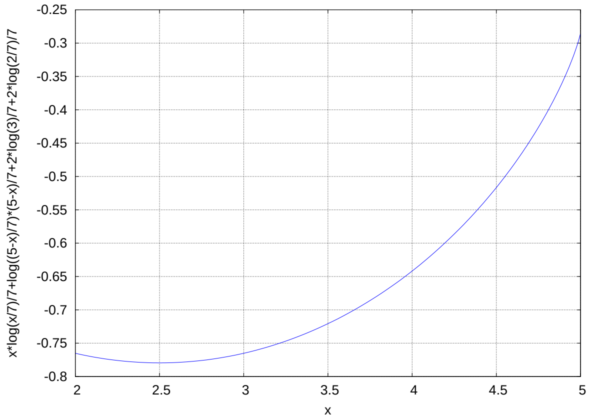

Fig. 1. The graph of $D(\theta_1(x))$

5.3. *Another sub-class of circulant states*

For each $\varepsilon > 0$, we define the map $\phi_\varepsilon : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ as

$$\phi_\varepsilon = \frac{1}{\Lambda}\phi_0 + \frac{\varepsilon}{\Lambda}\phi_1 + \frac{\varepsilon^{-1}}{\Lambda}\phi_2,$$

where $\Lambda = 1 + \varepsilon + \varepsilon^{-1}$.

One can find that the marginal states to be the same as Equation (5.8) and the state $\theta_2(\varepsilon)$ obtained from ϕ_ε is given by

$$\theta_2(\varepsilon) = \sum_{i,j}\phi_\varepsilon\left(e_{ji}\right)\otimes e_{ij} = \frac{1}{3\Lambda}\sum_{i,j} e_{ij}\otimes e_{ij} + \frac{\varepsilon}{3\Lambda}\left(e_{00}\otimes e_{11} + e_{11}\otimes e_{22} + e_{22}\otimes e_{00}\right) + \frac{\varepsilon^{-1}}{3\Lambda}\left(e_{00}\otimes e_{22} + e_{11}\otimes e_{00} + e_{22}\otimes e_{11}\right). \tag{5.11}$$

Notice that $\theta_2(\varepsilon)$ has a same circulant structure with $\theta_1(\alpha)$, i.e. it is given by the mixture of ϕ_0, ϕ_1 and ϕ_2. However the components of coefficient are different.

We get 0, $\frac{1}{\Lambda}$, $\frac{\varepsilon}{\Lambda}$ and $\frac{\varepsilon^{-1}}{\Lambda}$ for the eigenvalues of $\theta_2(\varepsilon)$. Therefore, we have

$$D(\theta_2(\varepsilon)) = \frac{1}{1+\varepsilon+1/\varepsilon}\log\frac{1}{1+\varepsilon+1/\varepsilon} + \frac{\varepsilon}{1+\varepsilon+1/\varepsilon}\log\frac{\varepsilon}{1+\varepsilon+1/\varepsilon}$$

$$+\frac{1/\varepsilon}{1+\varepsilon+1/\varepsilon}\log\frac{1/\varepsilon}{1+\varepsilon+1/\varepsilon}+\frac{1}{1+\varepsilon+1/\varepsilon}\log 3.$$

The following theorem gives us a useful characterization of $\theta_2(\varepsilon)$.[28]
Theorem 5.2 *The states of $\theta_1(\varepsilon)$ are classified by ε as follows:*
1. *$\theta_1(\varepsilon)$ is separable if $\varepsilon = 1$;*
2. *$\theta_1(\varepsilon)$ is both PPT and entangled for $\varepsilon \neq 1$.*

The graph of $D(\theta_2(\varepsilon))$ is shown in Fig. 2. $D(\theta_2(\varepsilon))$ takes the minimal value at $\varepsilon = 1$ and it is approximated about $D(\theta_2(1)) = -\frac{2}{3}\log 3 \approx -0.7324$. As $\varepsilon \to 0$ or ∞, $\theta_2(\varepsilon)$ converges to a separable perfectly correlated state which can be recognized as a "classical state"

$$\lim_{\varepsilon\to 0}\theta_2(\varepsilon) = \frac{1}{3}\left(e_{00}\otimes e_{22}+e_{11}\otimes e_{00}+e_{22}\otimes e_{11}\right), \tag{5.12}$$

$$\lim_{\varepsilon\to \infty}\theta_2(\varepsilon) = \frac{1}{3}\left(e_{00}\otimes e_{11}+e_{11}\otimes e_{22}+e_{22}\otimes e_{00}\right), \tag{5.13}$$

and for every $\varepsilon > 0$,

$$D(\theta_2(\varepsilon)) < 0 = \lim_{\varepsilon\to 0} D(\theta_2(\varepsilon)) = \lim_{\varepsilon\to\infty} D(\theta_2(\varepsilon)). \tag{5.14}$$

It shows that a correlation of a PPT entangled state θ_2 $(\varepsilon \neq 1)$ is weaker than that of the (classical) separable perfectly correlated states in the sense of (4.16).

Now, since $\theta_1(\alpha)$ and $\theta_2(\varepsilon)$ have common marginal states, we can compare the order of quantum correlations for them. One has, for example,

$$D(\theta_2(1)) \approx -0.7324 > -0.7587 \approx D(\theta_1(3.1)). \tag{5.15}$$

Accordingly Theorem 5.1 and 5.2, however, $\theta_2(1)$ is separable while $\theta_1(3.1)$ is entangled state. Incidentally, this means that the correlation for the separable state $\theta_2(1)$ is stronger than the entangled state $\theta_1(3.1)$ in the sense of (4.16).

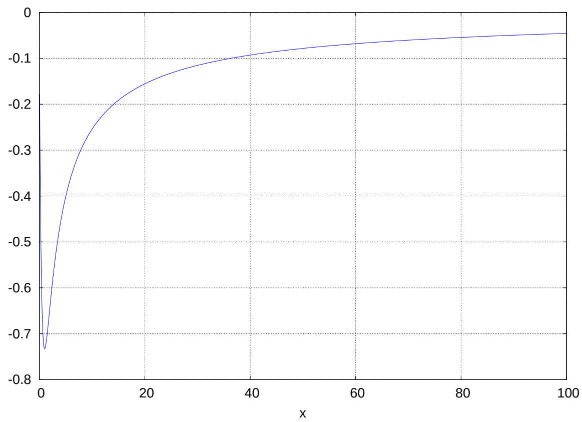

Fig. 2. The graph of $D(\theta_2(x))$

§6. Correlations of Bell diagonal states

In this section, we observe the geometric model studied by B. Baumgartner, B.C. Hiesmayr and H. Narnhofer.[8] We set

$$\Omega_{0,0} = \frac{1}{\sqrt{3}}\sum_{i=0}^{2} e_i\otimes e_i \in (\mathbb{C}^3)^{\otimes 2}.$$

and define $\Omega_{k,\ell}$, for any k,ℓ $(0 \le k,\ell \le 2)$, as

$$\Omega_{k,\ell} = (W_{k,\ell}\otimes I_3)\Omega_{0,0},$$

where $W_{k,\ell}$ means the circle action given by

$$e_k \mapsto \exp\Big(\frac{2\pi i}{3}(k-\ell)\Big)\, e_{k-\ell}, \quad (i = 0, 1, 2).$$

Let us define the linear operator $\theta(\alpha, \beta)$ on $\mathbf{B}(\mathbb{C}^3 \otimes \mathbb{C}^3)$ as

$$\begin{aligned}
\theta(\alpha,\beta) := \frac{1-\alpha-\beta}{9} I_3 \otimes I_3 + \alpha|\Omega_{0,0}\rangle\langle\Omega_{0,0}| + \frac{\beta}{2}\Big(|\Omega_{1,0}\rangle\langle\Omega_{1,0}| + |\Omega_{2,0}\rangle\langle\Omega_{2,0}|\Big) \\
= \frac{1+2(\alpha+\beta)}{9} \sum_{i=0}^{2} e_{ii} \otimes e_{ii} \\
+ \frac{1-\alpha-\beta}{9}(e_{00} \otimes e_{11} + e_{00} \otimes e_{22} + e_{11} \otimes e_{00} + e_{11} \otimes e_{22} + e_{22} \otimes e_{00} + e_{22} \otimes e_{11}) \\
+ \frac{2\alpha-\beta}{6}(e_{01} \otimes e_{01} + e_{02} \otimes e_{02} + e_{10} \otimes e_{10} + e_{12} \otimes e_{12} + e_{20} \otimes e_{20} + e_{21} \otimes e_{21}).
\end{aligned} \tag{6.1}$$

By computing the eigenvalues for $\theta(\alpha,\beta)$, one can find that $\theta(\alpha, \beta)$ is a state if and only if the parameters α, β satisfy

$$\alpha + \beta \leq 1, \quad 2\alpha - 7\beta \leq 2, \quad -8\alpha + \beta \leq 1. \tag{6.2}$$

Moreover, we ask for the condition that $\theta(\alpha,\beta)$ can be positive under partial transpose. The partial transpose ${}^T\theta(\alpha, \beta)$ of $\theta(\alpha,\beta)$ is given by

$$\begin{aligned}
{}^T\theta(\alpha, \beta) = \frac{1+2(\alpha+\beta)}{9} \sum_{i=0}^{2} e_{ii} \otimes e_{ii} \\
+ \frac{1-\alpha-\beta}{9}(e_{00} \otimes e_{11} + e_{00} \otimes e_{22} + e_{11} \otimes e_{00} + e_{11} \otimes e_{22} + e_{22} \otimes e_{00} + e_{22} \otimes e_{11}) \\
+ \frac{2\alpha-\beta}{6}(e_{01} \otimes e_{10} + e_{02} \otimes e_{20} + e_{10} \otimes e_{01} + e_{12} \otimes e_{21} + e_{20} \otimes e_{02} + e_{21} \otimes e_{12}).
\end{aligned} \tag{6.3}$$

Accordingly, the PPT condition holds for α, β which satisfy

$$\alpha + \beta \geq -\frac{1}{2}, \quad 8\alpha - \beta \leq 2, \quad -4\alpha + 5\beta \leq 2. \tag{6.4}$$

On the domain which consists of α, β satisfying both (6.2) and (6.4), the state is either separable or bound entangled. However, it was shown that for this class all PPT states are separable (see[8]). For example, let us consider the line $\beta = 3/5$. One has: $\theta(\alpha, 3/5)$ is entangled if and only if $\alpha \in [0, 1/4) \cup (13/40, 2/5]$, and it is separable if and only if $\alpha \in [1/4, 13/40]$.

The marginal states ρ and σ are calculated to be the same as (5.8) . Hence we obtain

$$\begin{aligned}
D\Big(\theta(\alpha, 3/5)\Big) = -\frac{10\alpha-4}{15}\log\left(\frac{2-5\alpha}{45}\right) + \frac{31-10\alpha}{45}\log\left(\frac{31-10\alpha}{90}\right) \\
+ \frac{40\alpha+2}{45}\log\left(\frac{40\alpha+2}{45}\right) + \log 3.
\end{aligned} \tag{6.5}$$

By a simple calculation, it easily be verified that $D(\theta(\alpha, 3/5))$ is monotonically increasing if $\alpha \in [1/20, 2/5)$. Hence, the values of D for $\alpha \in [1/20, 1/4)$ (i.e. $\theta(\alpha, 3/5)$ is entangled.) are less than the ones for $\alpha \in [1/4, 13/40)$ (i.e. $\theta(\alpha, 3/5)$ is separable.) (see Fig. 3).

Next, we analyze $D(\theta(\alpha,\beta))$ on the line $\alpha = 0$. It is calculated to be

$$D\Big(\theta(0, \beta)\Big) = -\frac{7\beta-7}{9}\log\left(\frac{1-\beta}{9}\right) + \frac{7\beta+2}{9}\log\left(\frac{7\beta+2}{18}\right) + \log 3. \tag{6.6}$$

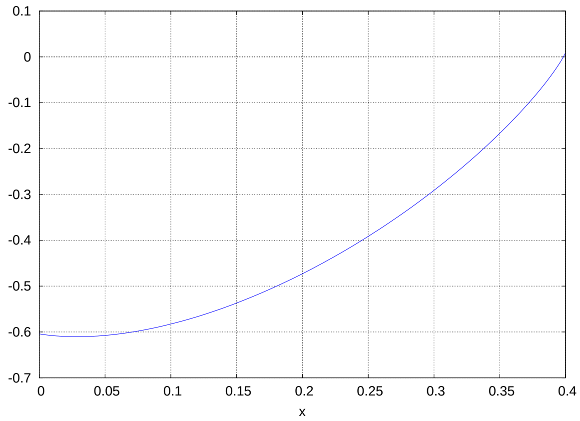

Fig. 3. The graph of $D(\theta(x, 3/5))$

Here, we obtain

$$D\big(\theta(0,\ 2/5)\big) = \log 5 - \frac{16}{15}\log 2 > \log\frac{5}{4} > 0, \tag{6.7}$$

$$D(\theta(0,1)) = \log\frac{2}{3} < 0. \tag{6.8}$$

It follows from the Mean Value Theorem that $D(\theta(0,\ \beta)) = 0$ has at least one solution. We can show easily that the function $D(\theta(0,\ \beta))$ is monotonically increasing (see Fig.4). Hence, it has just only one solution β_0 in the domain $[2/5,\ 1]$. This means that the state $\theta(0,\beta)$ has a non classical correlation when $\beta \in (\beta_0, 1]$, i.e. the quantum mutual entropy of $\theta(0,\beta)$ does not satisfy the Shannon's fundamental inequality.

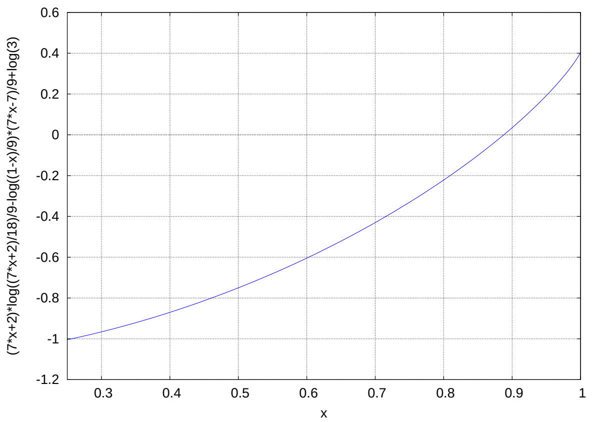

Fig. 4. The graph of $D(\theta(0, x))$

On the other hand, let us consider $D(\theta(\alpha,\beta))$ on the line $\beta = 0$. It is given by

$$D\big(\theta(\alpha,\ 0)\big) = -\frac{2\alpha-2}{3}\log\left(\frac{1-\alpha}{9}\right) - \frac{2\alpha-2}{9}\log\left(\frac{-2\alpha+2}{18}\right) + \frac{8\alpha+1}{9}\log\left(\frac{8\alpha+1}{9}\right) + \log 3. \tag{6.9}$$

In the way similar to the case of $D(\theta(0,\ \beta))$, we can find that there exists a solution α_0 in the interval $[1/4,\ 1]$. That is, the state $\theta(\alpha, 0)$ also has a non classical correlation on $(\alpha_0, 1]$ (see Fig. 5).

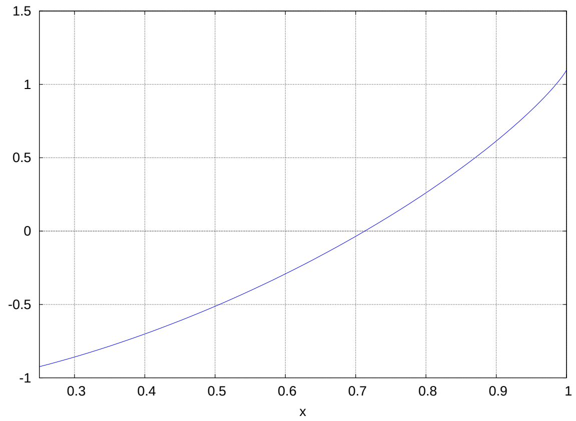

Fig. 5. The graph of $D(\theta(x, 0))$

§7. Summary and Discussion

In the section 2 we briefly reviewed the classical information theory and we mentioned that the mutual entropy is a proper criterion to measure a correlation of composite system as a transmitted information through a channel. Such interpretation is based on the fundamental probability relation (2.1) and also the Shannon's fundamental inequality (2.11). In the section 3 we drew a quantum analogue (3.14) of the fundamental relation (2.1) as the relation between a compound density θ_ϕ and a quantum conditional probability operator π_ϕ via an entanglement map ϕ, and we called it the quantum density relation. However we have not yet discussed the role of the relation (3.14) in the context of quantum information theory. Before the discussion we summarize the results of the section 4-6. The quantum mutual entropy $I_\phi(\rho, \sigma)$ is given by the relative entropy between the density θ_ϕ itself and its marginal product state $\rho \otimes \sigma$ as an extension of the classical definition (2.2). The conditional entropies $S_\phi(\sigma \mid \rho)$ and $S_\phi(\rho \mid \sigma)$ also are defined as an quantum analogue by using the classical relation (2.6) and (2.7). Such entropies have non-classical properties (see Example 4.5). How can we understand such non-classical properties? Due to the non-classical properties we can classify a subset of $\mathbf{S}_{\mathrm{ENT}}$ (the set of entangled states) by using D quantity (see Proposition 4.7 and 4.8. Also see Figure 4 and 5). This is, in some sense, one of the answers above question. The non-classical properties of such entropies are found on the subset of entangled states, however such properties do not give the necessary condition of entangled states. Anyway it is natural to think that the mutual entropy $I_\phi(\rho, \sigma)$ of entangled state represents the correlation of entanglement. From this point of view during the last one decade there have been a lot of studies of quantum correlation by means of the mutual entropy.[2),3),11),12),14),21),22),23),33),37),42)] On the base of above observation we introduced the order of quantum correlation in the sense of (4.16). However we know that there exist quantum entangled states with weaker correlations than some separable states (see the inequality (5.15) and Figure 3 .). How can we understand such correlation between separable and entangled states on the base of the fundamental density relation (3.14)?

In the classical information theory the mutual entropy is the criterion to measure the transmitted information as mentioned above. We can also represent the quantum mutual entropy by channel representation using the relation (3.14). If a QCPO π is given, then we can define a channel Λ^* for any input state ρ_{in} by

$$\rho_{\mathrm{out}} = \Lambda^*(\rho_{\mathrm{in}}) := \mathrm{Tr}_{\mathcal{H}}(\rho_{\mathrm{in}}^{\frac{1}{2}} \otimes 1)\,\pi\,(\rho_{\mathrm{in}}^{\frac{1}{2}} \otimes 1). \tag{7.1}$$

In this scheme we can call π a channel density. Applying this scheme to the density operator θ_ϕ, we have

$$I_\phi(\rho, \sigma) = \mathrm{Tr}\theta_\phi\,(\log\theta_\phi - \log(\rho \otimes \sigma))$$

$$= \mathrm{Tr}(\rho^{\frac{1}{2}} \otimes 1)\, \pi_\phi\, (\rho^{\frac{1}{2}} \otimes 1) \left(\log(\rho^{\frac{1}{2}} \otimes 1)\pi_\phi(\rho^{\frac{1}{2}} \otimes 1) - \log(\rho \otimes \Lambda_\phi^* \rho) \right)$$
$$=: I_\phi(\rho,\, \Lambda_\phi^*), \tag{7.2}$$

where $\Lambda_\phi^*(\rho) := \mathrm{Tr}_{\mathcal{H}}(\rho^{\frac{1}{2}} \otimes 1)\pi_\phi(\rho^{\frac{1}{2}} \otimes 1)$. The above representation of $I_\phi(\rho,\, \Lambda_\phi^*)$ may be an quantum analogue of the classical representation (2.8). However this mutual entropy $I_\phi(\rho,\, \Lambda_\phi^*)$ does not satisfy the following inequality on the subset of entangled state (see Example 4.5. Also see Figure 4 and 5):

$$0 \le I_\phi(\rho,\, \Lambda_\phi^*) \le \min\left\{ S(\rho),\, S(\Lambda_\phi^* \rho) \right\}. \tag{7.3}$$

From this point of view the quantum mutual entropy $I_\phi(\rho,\, \sigma)$ is not a proper measure of the transmitted information through the channel.

Now we have another question from the information theoretical point of view: how can we connect $I_\phi(\rho,\, \sigma)$ to the transmitted information through the entanglement channel Λ_ϕ^* on the base of the quantum density relation (3.14)? Notice that the entanglement channel Λ_ϕ^* can be recognized as the channel which transmits information from a quantum input to a quantum output. The measure of transmitted information for such *q*-*q* channel, which satisfies the inequality (7.3), was introduced by M. O..[34),35)] Using his measure, we can show that $I_\phi(\rho,\, \sigma)$ can be divided into two types of correlation, one of such correlations is a transmitted correlation and another one may be corresponding to non-classical correlation. We will study the detail of the above problems in the forthcoming paper.

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